THE EXPECTED VALUE

1 Calculate the expected value and the standard deviation when rolling a die. Plot the probability distribution d.

Let X be the number of pips.

$\Rightarrow \quad \Omega = \{1, 2, \dots, 6\}$

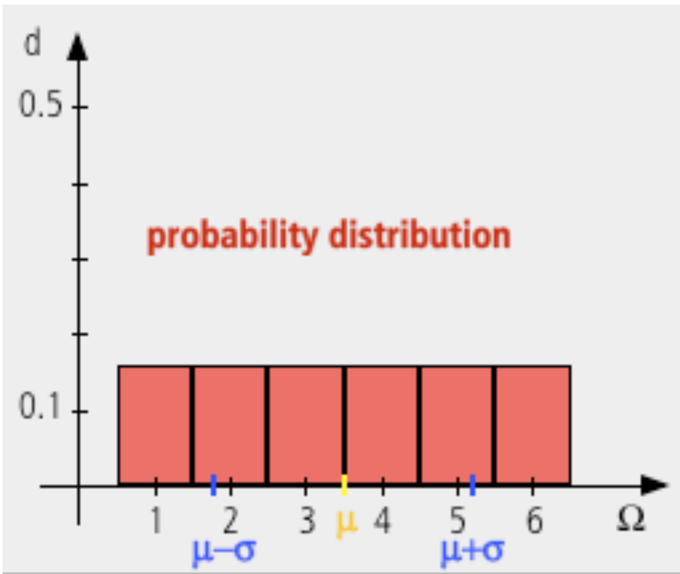
X is a random variable.

$p(1) = p(2) = \dots = p(6) = \frac{1}{6}$

$\Rightarrow \quad d(1) = \dots = d(6) = \frac{1}{6}$

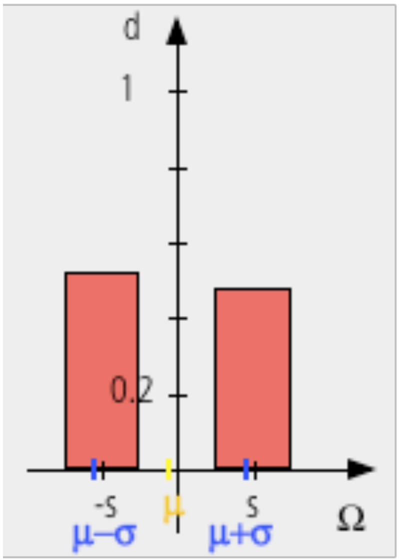
$\Rightarrow \quad \mu = \sum_{i=1}^{6} \frac{1}{6} \cdot i = \mathbf{3.5}$

$\Rightarrow \quad \sigma = \sqrt{\sum_{i=1}^{6} \frac{1}{6} \cdot \left(\frac{7}{2} - i\right)^2} \approx \mathbf{1.70}$



2 Calculate the expected value and the standard deviation at roulette when
a) putting on colour.
b) putting on carré (four numbers arranged in a square on the tableau).
c) putting on a single number.

Plot the probability distribution each time.
Which strategy holds the biggest risk?



a) Let s be the stake. Then s can be won or lost.

$\Rightarrow \quad \Omega = \{-s, s\}$

$$d(-s) = p(-s) = \frac{19}{37}$$

$$d(s) = p(s) = \frac{18}{37}$$

$$\Rightarrow \quad \mu = \frac{19}{37}\cdot(-s) + \frac{18}{37}\cdot s = \mathbf{-\frac{1}{37}s}$$

$$\Rightarrow \quad \sigma = \sqrt{\frac{19}{37}\cdot\left(-s-\left(-\frac{1}{37}s\right)\right)^2 + \frac{18}{37}\cdot\left(s-\left(-\frac{1}{37}s\right)\right)^2} \approx \mathbf{0.9996s}$$

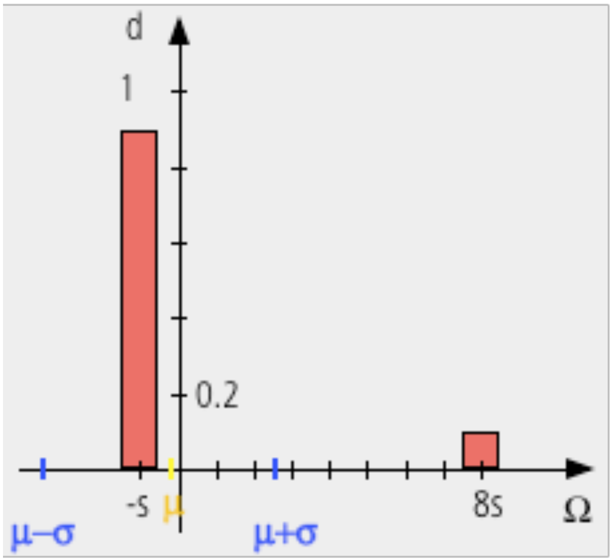
b) Let s be the stake. Then 8s can be won or s can be lost.

$\Rightarrow \quad \Omega = \{-s, 8s\}$

$p(-s) = \frac{33}{37}$

$p(8e) = \frac{4}{37}$

$$\Rightarrow \quad \mu = \frac{33}{37}\cdot(-s) + \frac{4}{37}\cdot 8s = \mathbf{-\frac{1}{37}s}$$



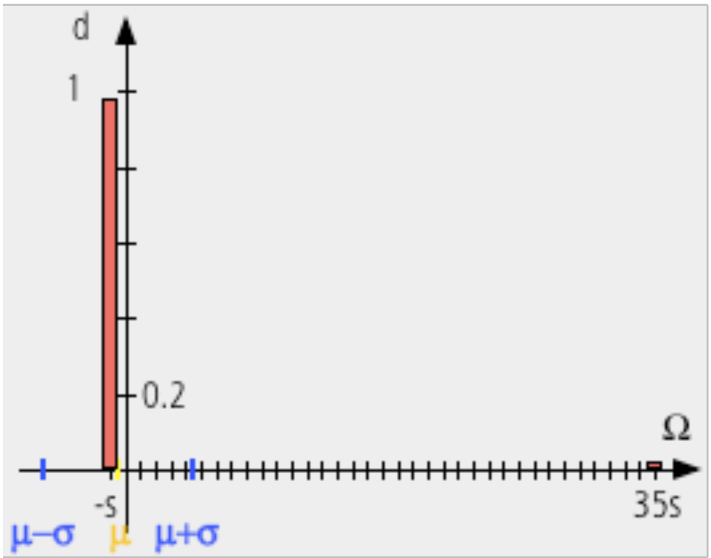
$$\Rightarrow \quad \sigma = \sqrt{\frac{33}{37}\cdot\left(-s-\left(-\frac{1}{37}s\right)\right)^2 + \frac{4}{37}\cdot\left(8s-\left(-\frac{1}{37}s\right)\right)^2} \approx \mathbf{2.7947s}$$

c) Let s be the stake. Then 35s can be won or s can be lost.

$\Rightarrow \quad \Omega = \{-s, 35s\}$

$p(-s) = \frac{36}{37}$

$p(35s) = \frac{1}{37}$



$$\Rightarrow \quad \mu = \frac{36}{37}\cdot(-s) + \frac{1}{37}\cdot 35s = \mathbf{-\frac{1}{37}e}$$

$$\Rightarrow \quad \sigma = \sqrt{\frac{36}{37}\cdot\left(-s-\left(-\frac{1}{37}s\right)\right)^2 + \frac{1}{37}\cdot\left(35s-\left(-\frac{1}{37}s\right)\right)^2} \approx \mathbf{5.8378s}$$

The biggest risk holds the strategy "putting on a single number".

3 Calculate the premium for a 63 year old woman who wants to take out a life insurance over CHF 500,000 for two years under the same conditions as in the 2nd example.

From the point of view of the insurance company it is a probability experiment with $\Omega = \{\text{P, P} - 500{,}000\}$.

$$p(P) = p(\text{"woman survives"}) = 0.993932 \cdot 0.993404 = 0.98737602$$

$$p(P - 500{,}000) = p(\text{"woman dies"}) = 1 - p(P) = 0.01262398$$

$$\Rightarrow \quad \mu = 0.987376 \cdot P + 0.01262398 \cdot (P - 500{,}000) = 0$$

$$\Leftrightarrow \quad P = 0.01262398 \cdot 500{,}000 = 6{,}311.99$$

A fair premium would be **CHF 6,311.99**.

4 Calculate the bank's fair offer in the game show DEAL OR NO DEAL if there are the amounts of 250,000 and 150,000 and 30,000 and 10,000 and 5 francs still in the game.

From the point of view of the player it is a probability experiment with $\Omega = \{250{,}000,\ 150{,}000,\ 30{,}000,\ 10{,}000,\ 5\}$.

$$p(250{,}000) = p(150{,}000) = p(30{,}000) = p(10{,}000) = p(5) = \frac{1}{5}$$

$$\Rightarrow \quad \mu = \frac{1}{5} \cdot 250{,}000 + \frac{1}{5} \cdot 150{,}000 + \frac{1}{5} \cdot 30{,}000 + \frac{1}{5} \cdot 10{,}000 + \frac{1}{5} \cdot 5 = \frac{440{,}005}{5} = 88{,}001$$

A fair offer would be **CHF 88,001.–**.

5 A pair of dice are rolled. If this was done many times what would the average sum of pips be?

$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

$p(2) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$ $\quad p(6) = 5\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{5}{36}$ $\quad p(10) = 3\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{3}{36}$

$p(3) = 2\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{2}{36}$ $\quad p(7) = 6\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{6}{36}$ $\quad p(11) = 2\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{2}{36}$

$p(4) = 3\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{3}{36}$ $\quad p(8) = 5\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{5}{36}$ $\quad p(12) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$

$p(5) = 4\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{4}{36}$ $\quad p(9) = 4\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{4}{36}$

$$\Rightarrow \quad \mu = \frac{1}{36}\cdot 2 + \frac{2}{36}\cdot 3 + \frac{3}{36}\cdot 4 + \frac{4}{36}\cdot 5 + \frac{5}{36}\cdot 6 + \frac{6}{36}\cdot 7$$
$$+ \frac{5}{36}\cdot 8 + \frac{4}{36}\cdot 9 + \frac{3}{36}\cdot 10 + \frac{2}{36}\cdot 11 + \frac{1}{36}\cdot 12 = \frac{252}{36} = 7$$

The average sum of the pips would be **7**.

6 A friend offers a game: he tosses two coins of 1 franc, you toss one coin of 2 francs. The one with more "heads" wins all the coins.

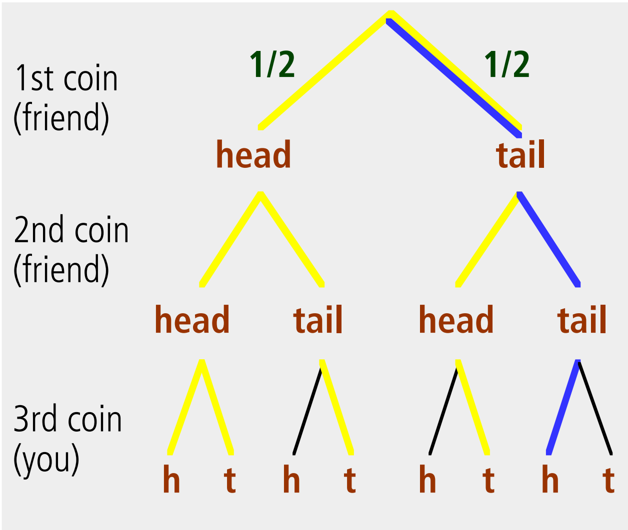
a) Is the game fair?
b) Who takes the bigger risk?
c) Plot the corresponding probability distributions.

From your point of view you can win 2 francs and lose 2 francs.

$\Omega = \{-2, 2\}$

$p(-2) = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$

$p(2) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$



a) $\mu = \frac{1}{2} \cdot (-2) + \frac{1}{8} \cdot 2 = -\frac{3}{4}$

The game is **unfair**, it is biased in favour of your friend. He wins CHF –.75 on average per game.

b) From your point if view:

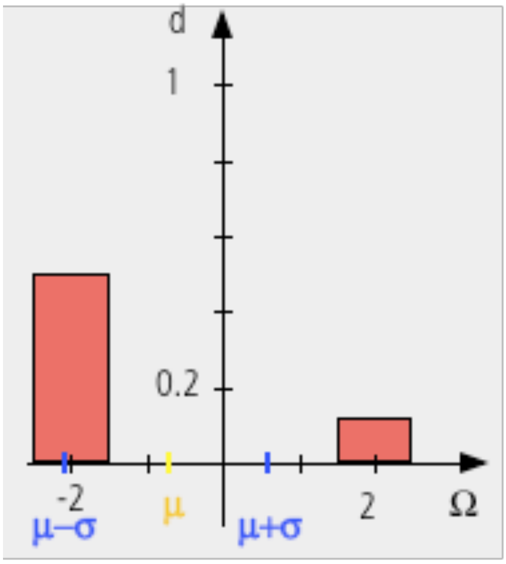
$$\sigma = \sqrt{\frac{1}{2} \cdot \left(-2 - \left(-\frac{3}{4}\right)\right)^2 + \frac{1}{8} \cdot \left(2 - \left(-\frac{3}{4}\right)\right)^2} \approx 1.314$$

From your friend's point if view is $\mu = \frac{1}{2} \cdot 2 + \frac{1}{8} \cdot (-2) = \frac{3}{4}$ and therefore

$$\sigma = \sqrt{\frac{1}{2} \cdot \left(2 - \frac{3}{4}\right)^2 + \frac{1}{8} \cdot \left(-2 - \frac{3}{4}\right)^2} \approx 1.314$$

The risk is the same.

c) The probability distribution for yourself:



The probability distribution for your friend: